VARIATIONAL PROBLEMS OF OPTIMIZATION OF CONTROL PROCESSES WITH FUNCTIONALS DEPENDING ON INTERMEDIATE VALUES OF THE COORDINATES

(VARIATSIONNYE ZADACHI OPTIMIZATSII PROTSESSOV UPRAVLENIIA S FUNKTSIONALAMI, ZAVISIASHCHIMI OT PROMEZHUTOCHNYKH ZNACHENII KOORDINAT)

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In the study of optimal processes one usually considers problems [1] for which the variational formulation leads to problems related to those of Lagrange, Mayer and Bolza in the calculus of variation [2,3,4]. Their functionals may depend on the values of the coordinates of the endpoints of the time interval under consideration.

In this work variational problems on the optimization of control processes in which the functionals depend on the values of the coordinates at interior points of this interval are considered.

1. Formulation of the problem. It is assumed that the behavior of the optimizing system is described by a system of ordinary differential equations of the following type

$$g_s = \dot{x}_s - f_s(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = 0 \qquad (s = 1, \ldots, n) \tag{1.1}$$

It is also assumed that the functions $u_k(t)$, $(k = 1, \ldots, m)$ are connected by r finite linear relations

$$\psi_k = \psi_k(u_1, \ldots, u_m, t) = 0 \qquad (k = 1, \ldots, r) \tag{1.2}$$

The variables $x_s(t)$ will be called coordinates, while $u_k(t)$ will be referred to as control parameters [1].

With the aid of the relations (1.2) one can perform a transition from closed to open regions of admissible variations of the control parameters. The details of this transformation are given in [5,6].

In what follows, it will be assumed that in the $(n + m + 1)$-dimensional space of the variables $x_1, \ldots, x_m, u_1, \ldots, u_m$, and t there exists an open region R_1 of admissible variations of the coordinates $x_s(t)$, of the control parameters $u_k(t)$, and of the time t; in this region the functions f_s and ψ_k are continuous, and possess continuous partial derivatives up to, and including, the third order [4].

Let us consider the interval $t_0 \leqslant t \leqslant t_{q+1} = T$, belonging to the region R_1 and let us concentrate our attention on q interior points $t = t_i$ $(i = 1, \ldots, q)$ for which the following relations hold:

$$\varphi_l\,[x(t_0), t_0, x(t_1), t_1, \ldots, x(t_q), t_q, x(T), T] = 0 \qquad (1.3)$$
$$(l = 1, \ldots, p \leqslant (n+1)(q+2) - 1)$$

For the sake of brevity, we have here denoted the set $x_1(t_i), \ldots, x_n(t_i)$ by the symbol $x(t_i)$.

We shall assume that the functions φ_l are continuous, and that they have continuous third order derivatives in the closed region R_2 of the $(n + 1) \times (q + 2)$ variables $x_1(t_0), \ldots, x_n(t_0), t_0$; $x_1(t_1), \ldots, x_n(t_1), t_1$; $\ldots, x_1(t_q), \ldots, x_n(t_q), t_q$ $x_1(T), \ldots, x_n(T), T$. We shall also assume that the matrix [4]

$$\left\| \frac{\partial\varphi_l}{\partial x_s(t_0)} \vdots \frac{\partial\varphi_l}{\partial t_0} \vdots \frac{\partial\varphi_l}{\partial x_s(t_1)} \vdots \frac{\partial\varphi_l}{\partial t_1} \vdots \cdots \vdots \frac{\partial\varphi_l}{\partial x_s(T)} \vdots \frac{\partial\varphi_l}{\partial T} \right\| \qquad (1.4)$$

is of rank p, equal to its number of rows. The elements of the lth row of this matrix are the partial derivatives of the function φ_l with respect to its arguments.

An element $(x, u, t) = (x_1, \ldots, x_n, u_1, \ldots, u_m, t)$ of a curve in the indicated $(n + m + 1)$-dimensional space will be admissible if it belongs to the region R_1. The set $(x(t_0), t_0, x(t_1), t_1, \ldots, x(T), T)$ will be called an end-element of the curve. It is determined by both the intermediate and the end values of the coordinates and time. If the end-element belongs to the region R_2 it too is called an admissible element.

Points of discontinuity of the control parameters $u_k(t)$ will be called corner points of the curve. Their number is assumed finite in the interval $t_0 \leqslant t \leqslant T$. A curve with a finite number of corner points in the interval $t_0 \leqslant t \leqslant T$, all elements (including the end-elements) of which are admissible, will be called an admissible curve.

The following problem of optimization of control will be studied.

It is required to select from among all admissible curves, satisfying (in the interval $t_0 \leqslant t \leqslant T$) the equations (1.1) and (1.2), and at the points $t = t_i$ $(i = 0, 1, \ldots, q + 1)$, the relations (1.3), that curve which makes the functional

$$J = g\,[x(t_0), t_0, x(t_1), t_1, \ldots, x(t_q), t_q, x(t_{q+1}), t_{q+1}] + \int_{t_0}^{T} f_0(x, u, t)\, dt \tag{1.5}$$

a minimum.

The arguments of the function g are the same values $x_s(t_i)$ of the coordinates $x_s(t)$ at the points $t = t_i$ which occur in the function φ_l. The problems on the maxima can be reduced to problems on the minima by a change of the sign of the functional. They will not be considered separately.

The formulated problem differs from that of Mayer and Bolza of the calculus of variation in that the quantities $x_s(t_i)$, $(s = 1, \ldots, n;$ $i = 1, \ldots, q)$ enter into the relations (1.3) and into the functional (1.5). They correspond to the points $t = t_i$ interior to interval $t_0 \leqslant t \leqslant T$.

Such problems arise in the construction of optimal controls when the criterion for an optimum is taken to be the deviation of the system from the equilibrium position at some fixed instant of time $t = t_1$, under conditions of the type [5,6]

$$\varphi_l\,[x(t_0), t_0, x(T), T] = 0$$

It is assumed here that $t_0 < t_1 < T$. The problem on the minimum (or maximum) of the deviation of a system, and many other problems, lead to an analogous formulation.

It should be noted that the Mayer-Bolza problem, considered in [6], is a particular case of the one described here, and that it can be obtained from the present one by setting $q = 0$.

2. Condition for the stationary state of the functional J. Let us consider a curve E satisfying the equations (1.1) and (1.2). We shall assume that on this curve the matrix [6]

$$\frac{\partial\psi}{\partial u} = \left\| \frac{\partial\psi_k}{\partial u_\beta} \right\|$$

is of rank r. (The element in the kth row and βth column is the partial derivative $\partial\psi_k / \partial u_\beta$.) Furthermore, we shall assume that on the given curve the following "conditions of non-tangency" hold

$$\frac{\partial \varphi_l}{\partial t_i} + \sum_{\alpha=1}^{n} \frac{\partial \varphi_l}{\partial x_\alpha(t_i)} \dot{x}_\alpha(t_i) \neq 0 \qquad (l = 1, \ldots, p) \tag{2.1}$$

Under these conditions, one can prove lemmas on the inclusion of the curve E in a one-parameter, or many-parameter, family of comparison curves satisfying the equations (1.1) and (1.2). The contents of the lemmas coincide almost with those given by Bliss [4], and they will not be reproduced here.

Let us suppose that there exists a curve E which makes the functional J a minimum. On the basis of a lemma on the inclusion, the curve can be included in a $(p + 1)$-parameter family of comparison curves, and it can be made to belong to this family when the parameter has the value zero. Forming the total differential of this family, and repeating arguments analogous to those given in [4], we can prove that the first necessary condition for the minimum of the functional J is fulfilled on this curve. This condition is known as the rule of multipliers, or the condition of stationary state for the functional J.

Introducing into the consideration the function [6]

$$L = f_0 + \sum_{s=1}^{n} \lambda_s g_s - \sum_{k=1}^{r} \mu_k \psi_k = \sum_{s=1}^{n} \lambda_s \dot{x}_s - H \tag{2.2}$$

$$\varphi = g + \sum_{l=1}^{p} \rho_l \varphi_l \tag{2.3}$$

where

$$H = H_\lambda + H_\mu = \sum_{s=0}^{n} \lambda_s f_s + \sum_{k=1}^{r} \mu_k \psi_k, \qquad (\lambda_0 = -1) \tag{2.4}$$

one can express this condition in the form of the equation $\Delta I = 0$, in which ΔI stands for the total variation of the functional I, which has the form

$$I = \varphi + \int_{t_0}^{T} L dt \tag{2.5}$$

The relation $\Delta I = 0$ must be satisfied on every curve which makes the functional J a minimum. The function $\lambda_s(t)$ $(s = 1, \ldots, n)$ and $\mu_k(t)$ $(k = 1, \ldots, r)$ cannot vanish simultaneously at any point on the interval $t_0 \leqslant t \leqslant T$.

Let us now try to obtain the explicit form of the necessary condition for the stationary state of the functional J, which is usually used in solving the problem on the optimization of control processes. In order to simplify the presentation, let us assume first of all that the interval $t_0 \leqslant t \leqslant T$ contains only one point $t = t_1$ where condition (1.3) is satisfied.

We shall assume that there are no points of discontinuity of the control parameters in this interval. Under these assumptions, the functional I can be expressed in the form

$$I = \varphi + \int_{t_0}^{t_1} L dt + \int_{t_1}^{T} L dt = \varphi + \int_{t_0}^{t_1} \Big[\sum_{s=1}^{n} \lambda_s \dot{x}_s - H\Big] dt + \int_{t_1}^{T} \Big[\sum_{s=1}^{n} \lambda_s \dot{x}_s - H\Big] dt \tag{2.6}$$

Forming its total variation ΔI, we obtain the expression [5]

$$\Delta I = \Delta\varphi - [\lambda_s (t_0)\, \dot{x}_s (t_0) - (H)_{t_0}]\, \delta t_0 + \tag{2.7}$$

$$+ [\lambda_s^- (t_1)\, \dot{x}_s^- (t_1) - \lambda_s^+ (t_1)\, \dot{x}_s^+ (t_1) - (H^-)_{t_1} + (H^+)_{t_1}]\, \delta t_1 +$$

$$+ [\lambda_s (T)\, \dot{x}_s (T) - (H)_T]\, \delta T + \int_{t_0}^{t_1} \Big[\sum_{s=1}^{n} \lambda_s \delta\dot{x}_s - \delta H\Big] dt + \int_{t_1}^{T} \Big[\sum_{s=1}^{n} \lambda_s \delta\dot{x}_s - \delta H\Big] dt$$

The superscripts minus and plus indicate here the left side and the right side limits of the functions involved; where no misunderstanding can arise, these symbols are omitted. Further, we have made use of the notation

$$\Delta\varphi = \sum_{s=1}^{n} \sum_{i=0}^{2} \frac{\partial \varphi}{\partial x_s (t_i)}\, \Delta x_s (t_i) + \sum_{i=0}^{2} \frac{\partial \varphi}{\partial t_i}\, \delta t_i =$$

$$= \sum_{i=0}^{2} \Big\{\sum_{s=1}^{n} \frac{\partial \varphi}{\partial x_s (t_i)}\, \delta x_s (t_i) + \Big[\sum_{s=1}^{n} \frac{\partial \varphi}{\partial x_s (t_i)}\, \dot{x}_s (t_i) + \frac{\partial \varphi}{\partial t_i}\Big] \delta t_i\Big\} \tag{2.8}$$

$$\delta H = \sum_{s=1}^{n} \frac{\partial H}{\partial x_s}\, \delta x_s + \sum_{k=1}^{m} \frac{\partial H}{\partial u_k}\, \delta u_k \tag{2.9}$$

The coefficients of δt_i on the right-hand side of the equation (2.7), can be transformed with the aid of the relation (2.4) into the form

$$\sum_{s=1}^{n} \lambda_s (t_i)\, \dot{x}_s (t_i) - (H)_{t_i} = (f_0)_{t_i}\, \delta t_i \tag{2.10}$$

Integration by parts yields

$$\int_{t_{i-1}}^{t_i} \sum_{s=1}^{n} \lambda_s \delta\dot{x}_s dt = \sum_{s=1}^{n} \lambda_s \delta x_s \Big|_{t_{i-1}}^{t_i} - \int_{t_{i-1}}^{t_i} \sum_{s=1}^{n} \dot{\lambda}_s \delta x_s dt \tag{2.11}$$

Substituting the expressions (2.8) to (2.11) into the relations (2.5) and (2.7), we obtain

$$\Delta I = \sum_{s=1}^{n}\left[\frac{\partial\varphi}{\partial x_s(t_0)} - \lambda_s(t_0)\right]\Delta x_s(t_0) + \sum_{s=1}^{n}\left[\frac{\partial\varphi}{\partial x_s(T)} + \lambda_s(T)\right]\Delta x_s(T) +$$
$$+\left[\frac{\partial\varphi}{\partial t_0} + (H)_{t_0}\right]\delta t_0 + \left[\frac{\partial\varphi}{\partial T} - (H)_T\right]\delta T + \left[\frac{\partial\varphi}{\partial t_1} - (H^-)_{t_1} + (H^+)_{t_1}\right]\delta t_1 +$$
$$+\sum_{s=1}^{n}\left[\frac{\partial\varphi}{\partial x_s(t_1)} + \lambda_s^-(t_1) - \lambda_s^+(t_1)\right]\Delta x_s(t_1) -$$
$$-\int_{t_0}^{t_1}\left[\sum_{s=1}^{n}\left(\dot\lambda_s + \frac{\partial H}{\partial x_s}\right)\delta x_s + \sum_{k=1}^{m}\frac{\partial H}{\partial u_k}\delta u_k\right]dt -$$
$$-\int_{t_1}^{T}\left[\sum_{s=1}^{n}\left(\dot\lambda_s + \frac{\partial H}{\partial x_s}\right)\delta x_s + \sum_{k=1}^{m}\frac{\partial H}{\partial u_k}\delta u_k\right]dt = 0 \qquad (2.12)$$

The analysis of this expression is analogous to the one described in [5] and leads to the following result. On a curve which makes the functional J a minimum, the following equations must hold

$$\dot\lambda_s + \frac{\partial H}{\partial x_s} = 0, \quad (s = 1, \ldots, n), \qquad \frac{\partial H}{\partial u_k} = 0 \quad (k = 1, \ldots, m) \qquad (2.13)$$

the end conditions

$$\lambda_s(t_0) - \frac{\partial\varphi}{\partial x_s(t_0)} = 0, \qquad \lambda_s(T) + \frac{\partial\varphi}{\partial x_s(T)} = 0 \quad (s = 1, \ldots, n) \qquad (2.14)$$

$$(H)_{t_0} + \frac{\partial\varphi}{\partial t_0} = 0, \qquad (H)_T - \frac{\partial\varphi}{\partial T} = 0 \qquad (2.15)$$

and the Erdmann-Weierstrass condition

$$\begin{gathered}\lambda_s^-(t_1) - \lambda_s^+(t_1) + \frac{\partial\varphi}{\partial x_s(t_1)} = 0 \qquad (s = 1, \ldots, n)\\ \frac{\partial\varphi}{\partial t_1} - (H^-)_{t_1} + (H^+)_{t_1} = 0\end{gathered} \qquad (2.16)$$

In the construction of the optimum control conditions one must also use the equations

$$\dot x_s - \frac{\partial H}{\partial \lambda_s} = 0 \quad (s = 1, \ldots, n), \qquad \frac{\partial H}{\partial \mu_k} = 0 \quad (k = 1, \ldots, r) \qquad (2.17)$$

which are equivalent to the equations (1.1) and (1.2), the relations (1.3), and the equations

$$x_s^-(t_1) - x_s^+(t_1) = 0, \qquad (s = 1, \ldots, n) \tag{2.18}$$

that reflect the continuity of the coordinates $x_s(t)$ at the point $t = t_1$.

We note that in this problem, just as in the Mayer-Bolza problem [6], one must make a distinction between the functions $x_s(t)$, $\lambda_s(t)$, $u_k(t)$ and $\mu_k(t)$, and the equations (2.14) and (2.17) in the distinct subintervals $t_0 \leqslant t \leqslant t_1$ and $t_1 \leqslant t \leqslant T$ of the interval $t_0 \leqslant t \leqslant T$. Let us agree to denote with a superscript minus the functions which belong to the first one of these intervals, and with a superscript plus the functions belonging to the second interval. Then we can obtain the following results.

For the determination of the $4n + 2m + 2r$ functions $x_s^{\pm}(t)$, $\lambda_s^{\pm}(t)$, $u_k^{\pm}(t)$, and $\mu_k^{\pm}(t)$ there exist the $2n + 2m$ equations (2.13), and the $2n + 2r$ equations (2.17). In the solution of the $4n$ differential equations there appear $4n$ arbitrary constants. For the purpose of finding them in addition to the p multipliers ρ_l and the quantities t_0, t_1 and T one can use the relations (1.3), (2.14), (2.15), (2.16) and (2.18). Their number is also equal to $4n + p + 3$.

Assuming that the interval $t_0 \leqslant t \leqslant T$ contains only one point, $t = t^*$, of discontinuity of the control parameters $u_k(t)$, and repeating the previous arguments, we can derive the equations (2.13) with the end conditions (2.14) and (2.15), and the Erdmann-Weierstrass conditions

$$\lambda_s^-(t^*) - \lambda_s^+(t^*) = 0 \qquad (s = 1, \ldots, n), \qquad (H^-)_{t^*} - (H^+)_{t^*} = 0 \tag{2.19}$$

If one assumes that there is only one point $t = t_1^*$ in the interval $t_0 \leqslant t \leqslant T$, at which the condition (1.3) is satisfied, and at which the control parameters are discontinuous, then one obtains the equations (2.13) and the relations (2.14) to (2.16).

Less restrictive assumptions in regard to the number of points of the type $t = t_i$, and of points of discontinuity of the control parameters $u_k(t)$, will not change the above listed results. They can, however, complicate the establishment of these results.

A comparison of the equations (2.16) and (2.19) shows that the points $t = t_i$ at which relation (1.3) holds, are quite different from the points $t = t^*$ where the control parameters $u_k(t)$ are discontinuous. At the first type of points there can occur a discontinuity of the multipliers $\lambda_s(t)$ and of the function H. At any point $t = t^*$, these functions are continuous.

The function H will be continuous at a point $t = t_i$ in that case when the quantity t_i does not appear in the relation (1.3) explicitly. If all the t_i ($i = 1, \ldots, q$) are absent from the relation (1.3), and if the time t does not enter explicitly in the functions f_s and ψ_k, then the equations of the problem have a first integral

$$H = h = \text{const} \tag{2.20}$$

The multipliers $\lambda_s(t)$ can have discontinuities; the function H will be continuous in the entire interval $t_0 \leqslant t \leqslant T$.

3. Necessary condition of Weierstrass. In the solution of the optimization problem of control processes, use is also made of the necessary condition of Weierstrass for a strict minimum of the functional J in addition to the condition of stationary state.

The proof of Weierstrass' condition is analogous to the one given in [6]. It will not be given here.

We shall need a generalization of the concept of normal state of a curve E which makes the functional J a minimum.

Here a curve is said to be normal if on it the determinant

$$\left|\frac{\partial \varphi_l}{\partial b_\alpha}\right| \qquad (l = 1, \ldots, p;\ \alpha = 1, \ldots, p)$$

is different from zero. This determinant is formed for the $(p + 1)$-parameter family of comparison curves used for the establishment of the necessary condition of Weierstrass for a strict minimum different from zero.

We shall assume that the curve E satisfies the necessary condition of Weierstrass for a strict minimum of the functional J, if on it there are satisfied the equations (1.1), (1.2) and the condition of stationary state with the multipliers $\lambda_s(t)$, $\mu_k(t)$ and ρ_l, and if the inequality

$$E \geqslant 0 \tag{3.1}$$

holds for these multipliers at each point of the curve for all possible admissible quantities $\dot{X}_s$, $U_k \neq \dot{x}_s$, u_k, connected by the equations (1.1) and (1.2).

Any normal curve E which yields a minimum of the functional J satisfies the necessary condition of Weierstrass.

The Weierstrass function E which occurs in the inequality (3.1) is determined by the formula

$$E = L(x, \dot{X}, U, \lambda, \mu, t) - L(x, \dot{x}, u, \lambda, \mu, t) - \sum_{s=1}^{n} \frac{\partial L}{\partial \dot{x}_s}(\dot{X}_s - \dot{x}_s) \tag{3.2}$$

in which x_s and u_k correspond to the curve which yields a minimum for J, while X_s and U_k are arbitrary admissible functions.

The relation (3.2) shows that the Weierstrass function E, as well as the function L, can have discontinuities of the first kind. Hence at the points of such discontinuities the inequality (3.1) has to be verified twice, once for the left side limit and once for the right side limits of the function E.

The substitution of the expression (2.2) into the relation (3.2) yields the formula

$$E = H(x, u, \lambda, \mu, t) - H(x, U, \lambda, \mu, t) \tag{3.3}$$

If one takes into consideration the identity $H_\mu \equiv 0$, the inequality (3.1) can be expressed in the form

$$H_\lambda(x_1, \ldots, x_n, u_1, \ldots, u_m, \lambda_1, \ldots, \lambda_n, t) \geqslant$$
$$\geqslant H_\lambda(x_1, \ldots, x_n, U_1, \ldots, U_m, \lambda_1, \ldots, \lambda_n, t) \tag{3.4}$$

This inequality is customarily used in solving problems of the optimization of controls.

The inequality (3.4) shows that the control parameters, which correspond to the optimum operating conditions, yield a maximum to the function H_λ. Hence, the Weierstrass condition necessary for a strict minimum of the functional J can be formulated in this problem in a form that is analogous to the principle of a maximum [1].

4. Example. The problem on the accumulation of periodic disturbances. As an example of the application of the described methods let us consider the following problem on the accumulation of periodic disturbances in a linear system with one degree of freedom [7].

We are given the second order equation

$$\ddot{x} + 2n\dot{x} + k^2 x = u \tag{4.1}$$

where $u(t)$ is the external disturbance satisfying the inequality

$$|u(t)| \leqslant U^* \tag{4.2}$$

It is required to find in the set of all functions $u(t)$, of period T_0, which satisfy the condition (4.2), a function which will assign a maximum to the function $x(t)$ in the stabilized state.

Let us introduce the notation

$$x_1 = x, \qquad x_2 = \dot{x} \tag{4.3}$$

and let us rewrite the equation (4.1) in the form

$$g_1 = \dot{x}_1 - x_2 = 0, \qquad g_2 = \dot{x}_2 + k^2 x_1 + 2n x_2 - u = 0 \tag{4.4}$$

Next we construct the relation

$$\psi = u^2 + v^2 - (U^*)^2 = 0 \tag{4.5}$$

which accomplishes the transition to the open region of the ranges of $u(t)$ and $v(t)$. Let us next introduce the periodicity

$$\varphi_1 = x_1(t_0) = 0, \qquad \varphi_2 = x_1(T) = 0, \qquad \varphi_3 = x_2(t_0) - x_2(T) = 0 \tag{4.6}$$

and the relation

$$\varphi_4 = T - t_0 - T_0 = 0 \tag{4.7}$$

which shows that the period is fixed.

The necessary condition for an extremum of $x_1(t)$ takes the form

$$\varphi_5 = x_2(t_1) = 0 \tag{4.8}$$

The functional J can be written in the form

$$J = -x_1(t_1) \tag{4.9}$$

The problem on the accumulation of the disturbances can be formulated in the following way.

One finds the functions which will make the functional (4.9) a minimum. These functions are found in the set of continuous functions $x_1(t)$ and $x_2(t)$ with a piece-wise continuous derivative $\dot{x}_2(t)$, and from the piece-wise continuous disturbances $u(t)$ satisfying in the interval $t_0 \leqslant t \leqslant T$ the equations (4.4) and (4.5), at its ends the relations (4.6) and (4.7), and at the point $t = t_1$, the relation (4.8).

In this form, this problem is a particular case of the one considered in the preceding sections. In solving it, one can make use of all the results established above.

Let us construct the functions H and φ. On the basis of the formulas (2.3) and (2.4) we have the following expressions for these functions

$$H = H_\lambda + H_\mu = \lambda_1 x_2 + \lambda_2 [-k^2 x_1 - 2n x_2 + u] + \mu [u^2 + v^2 - (U^*)^2] \tag{4.10}$$

$$\varphi = -x_1(t_1) + \rho_5 x_2(t_1) + \rho_1 x_1(t_0) + \rho_2 x_1(T) + \rho_3[x_2(t_0) - x_2(T)] + \rho_4(T - t_0 - T_0) \quad (4.11)$$

With the aid of the relations (2.18) we construct the equations

$$\dot{\lambda}_1 = k^2\lambda_2, \quad \dot{\lambda}_2 = -\lambda_1 + 2n\lambda_2, \quad \lambda_2 + 2\mu u = 0, \quad 2\mu v = 0 \quad (4.12)$$

Making use of the relations (2.14) and (2.15), we obtain the end conditions

$$\lambda_1(t_0) = \rho_1, \quad \lambda_1(T) = -\rho_2, \quad \lambda_2(t_0) = \rho_3, \quad \lambda_2(T) = \rho_3, \quad (H_\lambda)_{t_0} = (H_\lambda)_T = \rho_4 \quad (4.13)$$

The equations (2.16) lead to the following Erdmann-Weierstrass conditions:

$$\lambda_1^-(t_1) - \lambda_1^+(t_1) = 1, \quad \lambda_2^-(t_1) - \lambda_2^+(t_1) = -\rho_5, \quad (H_\lambda^-)_{t_1} - (H_\lambda^+)_{t_1} = 0 \quad (4.14)$$

For the point of discontinuity $t = t^*$ the function $u(t)$ will satisfy the equations

$$\lambda_1^-(t^*) - \lambda_1^+(t^*) = 0, \quad \lambda_2^-(t^*) - \lambda_2^+(t^*) = 0, \quad (H_\lambda^-)_{t^*} - (H_\lambda^+)_{t^*} = 0 \quad (4.15)$$

If the discontinuity of the control parameter occurs at the point $t = t_1^*$, where the relation (4.8) is satisfied, then one has to use the conditions (4.14).

The inequality (3.4) yields

$$\lambda_2 u \geqslant \lambda_2 U \quad (4.16)$$

Hence, we obtain the following values for $u(t)$:

$$u(t) = U^* \text{ for } \lambda_2 > 0, \qquad u(t) = -U^* \text{ for } \lambda_2 < 0 \quad (4.17)$$

The inequalities (4.12) and (4.17) show that the control $u(t)$ takes on only the boundary values $u(t) = \pm U^*$ at almost all points of the interval $t_0 \leqslant t \leqslant T$. The exceptions are the finite number of points $t = t^*$ at which $\lambda_2(t^*) = 0$.

On the basis of the relation (4.15) we see that the functions $\lambda_1(t)$, $\lambda_2(t)$, and H are continuous at $t = t^*$. The multiplier $\lambda_1(t)$ has a discontinuity at the point $t = t_1$. This is revealed by the first relation of (4.14). The second one of these conditions shows that the multiplier $\lambda_2(t)$ can also have a discontinuity at this point. The function H will be continuous at this point.

The condition on the continuity of the function H at the point $t = t_1$ leads to the relation

$$\frac{\lambda_2^-(t_1)}{\lambda_2^+(t_1)} = \frac{u^+(t_1) - k^2 x_1^+(t_1)}{u^-(t_1) - k^2 x_1^-(t_1)} \tag{4.18}$$

In case $T_0 \geqslant T_1 = 2\pi/k_1$, $k_1^2 = k^2 - n^2$ and $k > n$, one can find a function $u(t)$ which will satisfy the inequality

$$u_{\max} < k^2 x_{1\max}$$

Here, $u_{\max}$ and $x_{1\max}$ represent the largest values of $u(t)$ and $x_1(t)$. An example of this is the harmonic [8] disturbance $u(t) = u_{\max} \sin \omega t$. Hence, if $T_0 \geqslant T_1$, this inequality will be satisfied also by the optimal control. Therefore, the multiplier $\lambda_2(t)$ will not change sign at the point $t = t_1$. In this case, the control parameter will be continuous at the point $t = t_1$, as is shown by the formula (4.17). On the basis of formula (4.18) we find that the multiplier $\lambda_2(t)$ is continuous at this point.

Eliminating the multiplier $\lambda_1(t)$ from the relation (4.11), we obtain the following second order differential equation

$$\ddot{\lambda}_2 - 2n\dot{\lambda}_2 + k^2\lambda_2 = 0 \tag{4.19}$$

Its solution, satisfying the conditions (4.13), has the form

$$\lambda_2 = Ce^{nt} \sin (k_1 t + \alpha) \tag{4.20}$$

Here, α is determined by the formula

$$\tan\alpha = \frac{e^{nT_0} \sin k_1 T_0}{1 - e^{nT_0} \cos k_1 T_0} \tag{4.21}$$

It is constructed with the aid of the relation $\lambda_2(t_0) = \lambda_2(T)$, which is obtained from the second pair of the equations (4.13).

In the "resonance" case, $T_0 = T_1$, we find, on the basis of (4.20), that $\tan \alpha = 0$. Setting $\alpha = 0$, we obtain

$$\lambda_2(t) = Ce^{nt} \sin k_1 t$$

Then,

$$u(t) = U^* \quad \left(0 < t < \frac{\pi}{k_1}\right), \qquad u(t) = -U^* \quad \left(\frac{\pi}{k_1} < t < \frac{2\pi}{k_1}\right) \tag{4.22}$$

The value $\alpha = \pi$ corresponds to the second extremum of the function $x_1(t)$ in the interval $t_0 \leqslant t \leqslant T$.

When $T_0 = iT_1$, one finds again that $\tan\alpha = 0$. For the values $\alpha = j\pi$ $(j = 0, 1, \ldots)$, which are solutions of this equation, one finds control conditions which lead ultimately to the "resonance" case $T_0 = T_1$. Different values of α correspond to different points of the extremum of the function $x_1(t)$ in the interval $t_0 \leqslant t \leqslant T$.

When $T_0 < T_1$ one has to consider all the equations and conditions of the variational problem.

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